Positivity of Schur function expansions of Thom polynomials

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Abstract

Combining the approach to Thom polynomials via classifying spaces of singularities with the Fulton-Lazarsfeld theory of cone classes and positive polynomials for ample vector bundles, we show that the coefficients of the Schur function expansions of the Thom polynomials of stable singularities are nonnegative with positive sum.

1 Introduction

The global behavior of singularities[1] is governed by their *Thom polynomials* (cf. [15], [2], [8], [14]). As these polynomials are quite complicated even for "simplest" singularities, it is important to study their structure. There is a recent attempt to present Thom polynomials via their *Schur function expansions* (cf. [4], [11], [12]) instead of using the "traditional" basis of monomials in Chern classes.

In the present paper, we study the Schur function expansions of Thom polynomials from a "qualitive" point of view. Contrary to [14], [11], [12],

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[1]In the present paper we study complex singularities.

where the Szücs-Rimanyi approach via symmetries of singularities was used, we follow here the Kazarian approach [8] to Thom polynomials. In fact, both approaches rely on suitable "classifying spaces of singularities". We substitute the jet automorphism group by the group of the linear transformations $GL_m \times GL_n$. This allows one to extend the definition of Thom polynomials for the maps $f : M \to N$ of complex manifolds to pairs of vector bundles. It is convenient to pass to homotopy theory, where each pair of bundles can be pulled back from the universal pair of bundles on $BGL_m \times BGL_n$.

We apply the Fulton-Lazarsfeld theory of cone classes and positive polynomials for ample vector bundles [5], and deduce *nonnegativity* of the coefficients in the Schur function expansions of the Thom polynomials of the singularities stable under suspension, with positive sum.

This "positivity" was previously checked for a number of singularities: by Thom [15] for $A_1(r)$, by Feher and Komuves [4] for some second order Thom-Boardman singularities, by the first author [11], [12] for $I_{2,2}(r)$, $A_3(r)$, and for $A_i(r)$ under the aditional assumption that $\Sigma^j(f) = \emptyset$ for $j \geq 2$, by the first author and Ozturk for Thom polynomials from [14], by the second author for the Thom polynomials (from [8]) of singularities of functions, and by Ozturk [10] for $A_4(3)$, $A_4(4)$. [2] Some of these examples are listed in the last section.

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2 Thom polynomials

Fix $m, n, k \in \mathbf{N}$. We denote by Aut_n the group of k-jets of automorphisms of $(\mathbf{C}^n, 0)$, and by $\mathcal{J} = \mathcal{J}(m,n)$ the space of k-jets of functions $(\mathbf{C}^m, 0) \to (\mathbf{C}^n, 0)$ [3].

Moreover, we set

$$G := \mathrm{Aut}_m \times \mathrm{Aut}_n .$$

Consider the classifying principal G-bundle $EG \to BG$, i.e. a contractible space EG with a free action of the group G, and define

$$\widetilde{\mathcal{J}} = \widetilde{\mathcal{J}}(m,n) = EG \times_G \mathcal{J} .$$

Let $\Sigma \subset \mathcal{J}$ be an analytic closed G-invariant subset, which we shall call a "class of singularities". For a given class of singularities Σ, set

$$\widetilde{\Sigma} = EG \times_G \Sigma \subset \widetilde{\mathcal{J}}$$

[2] We use here the notation from [11], [12]. The calculations in the last three cases used extensively ACE [17].

[3] Though these objects depend on k, we omit "k" in the notation. This will happen also to other objects introduced later.

and denote by $\mathcal{T}^{\Sigma} \in H^{2\,\mathrm{codim}(\Sigma)}(\widetilde{\mathcal{J}}, \mathbf{Z})$ the dual class of $[\widetilde{\Sigma}]$ [4]. Since

$$H^{\bullet}(\widetilde{\mathcal{J}}, \mathbf{Z}) \simeq H^{\bullet}(BG, \mathbf{Z}) \simeq H^{\bullet}(BGL_m \times BGL_n, \mathbf{Z})\,,$$

$\mathcal{T}^{\Sigma}$ is identified with a polynomial in $c_1, \dots, c_m$ and $c'_1, \dots, c'_n$ which are the Chern classes of universal bundles R_m and R_n on BGL_m and BGL_n:

$$\mathcal{T}^{\Sigma} = \mathcal{T}^{\Sigma}(c_1, \dots, c_m, c'_1, \dots, c'_n)$$

(here, and in the following, we omit the pull back in the notation). This is a classical *Thom polynomial*. Speaking slightly informally, given a general map $f : M \to N$ of smooth varieties of corresponding dimensions m and n, the Thom polynomial

$$\mathcal{T}^{\Sigma}(c_1(M), \dots, c_m(M), c_1(N), \dots, c_n(N))$$

evaluates the dual class of the set where f has singularity "of the class Σ". A precise version of this statement is a content of the Thom theorem [15] (see also [8], Theorem 1 and [14], Sect. 6).

The *suspension*

$$\mathcal{S} : \mathcal{J}(m, n) \hookrightarrow \mathcal{J}(m+1, n+1)$$

allows one to increase the dimension of the source and the target simultaneously: with the local coordinates $x_1, x_2, \dots$ for the source and a function $f = f(x_1, \dots, x_m)$, the jet $(\mathcal{S}f) \in \mathcal{J}(m+1, n+1)$ is defined by

$$(\mathcal{S}f)(x_1, \dots, x_m, x_{m+1}) := (f(x_1, \dots, x_m), x_{m+1})\,.$$

Suppose that the class of singularities Σ is *stable under suspension.* By this we mean that it is a member $\Sigma_0 = \Sigma$ of a family

$$\{\Sigma_r \subset \mathcal{J}(m+r, n+r)\}_{r \geq 0}$$

such that

$$\Sigma_{r+1} \cap \mathcal{J}(m+r, n+r) = \Sigma_r$$

and

$$\mathcal{T}^{\Sigma_{r+1}}{}_{|H^{\bullet}(BGL_{m+r} \times BGL_{n+r}, \mathbf{Z})} = \mathcal{T}^{\Sigma_r}\,. \tag{1}$$

This means that if we specialize

$$c_{m+r+1} = c'_{n+r+1} = 0$$

[4] One may approximate $EG \to BG$ by a sequence of G-bundles over finite dimensional manifolds $(EG)_N \to (BG)_N$, where $N \to \infty$. Then "$\mathcal{T}^{\Sigma}$ is the dual class of $[\widetilde{\Sigma}]$" means that for any N the image of $\mathcal{T}^{\Sigma}$ in $H^{\bullet}((BG)_N, \mathbf{Z}) \cong H^{\bullet}((EG)_N \times_G \mathcal{J}, \mathbf{Z})$ is Poincaré dual to the class $[(EG)_N \times_G \Sigma]$ in $H^{BM}_{\bullet}((EG)_N \times_G \mathcal{J}, \mathbf{Z})$.

in the polynomial $\mathcal{T}^{\Sigma_{r+1}}$, we obtain the polynomial $\mathcal{T}^{\Sigma_r}$. If the class Σ is stable under V-equivalence (cf. [1, §I.6.5]) then it is stable in our sense.

The theorem of Thom has the following refinement due to Damon [3] for a class of singularities Σ which is stable under suspension: $\mathcal{T}^{\Sigma}$ is *supersymmetric*, i.e. is a polynomial in

$$(1 + c_1 + \ldots + c_m)/(1 + c_1' + \ldots + c_n') \quad \text{where} \quad i = 1, 2, \ldots .$$

In other words, for a general map $f : M \to N$,

$$\mathcal{T}^{\Sigma}(c_1(M), \ldots, c_m(M), c_1(N), \ldots, c_n(N))$$

is a polynomial in

$$c_i(TM - f^*TN) = [c(TM)/c(f^*TN)]_i \quad \text{where} \quad i = 1, 2, \ldots$$

(here, $TM - f^*TN$ is a virtual bundle). Cf. also [8, Theorem 2].

3 Schur functions expansions

Given a partition $I = (i_1, i_2, \ldots, i_l) \in \mathbf{N}^l$, where $0 \le i_1 \le i_2 \le \ldots \le i_l$, and vector bundles E and F on some variety X, the *Schur function*[5] $S_I(E - F)$ is defined by the following determinant:

$$S_I(E - F) := \Big| S_{i_p + p - q}(E - F) \Big|_{1 \le p, q \le l}, \tag{2}$$

where the entries are defined by the expression

$$\sum S_i(E - F) = \prod_b (1 - b) / \prod_a (1 - a) \,. \tag{3}$$

Here, the a's and b's are the Chern roots of E and F and the LHS of Eq. (3) is the *Segre class* of the virtual bundle $E - F$. So the Schur functions $S_I(E - F)$ lie in a ring containing the Chern classes of E and F; e.g., we can take the cohomology ring $H^{\bullet}(X, \mathbf{Z})$ or the Chow ring $A(X)$ for a smooth, algebraic X. More generally, the Schur functions $S_I(-)$ are well defined on the Grothendieck group $K_0(X)$ of vector bundles. In particular, given a vector bundle E and a partition I, we shall write respectively

$$S_I(E) \quad \text{and} \quad S_I(-E)$$

for $S_I(E - 0)$ and $S_I(0 - E)$, where 0 is the zero vector bundle.

[5] Usually this family of functions is called "super Schur functions" or "Schur functions in difference of bundles"; the classical Schur functions $\mathcal{S}_I$ will be used in Theorem 3 in the next section.

We have for any partition I,

$$S_I(E^* - F^*) = S_{I^\sim}(F - E) \tag{4}$$

where $I^\sim$ denotes the *dual partition* of I. In particular,

$$S_i(E^* - F^*) = c_i(F - E)$$

for any i, so that the Thom polynomial can be equivalently expressed as a polynomial in the

$$S_i(R_m^* - R_n^*)\text{'s}\,.$$

Here, and in the following, we omit pull back indices. Or, evaluated in the Chern classes $c_i(M)$, $c_j(N)$ of the manifolds involved in the map $f : M \to N$, it is expressed as a polynomial in the

$$S_i(TM^* - f^*TN^*)\text{'s}\,.$$

This convention was used in [11], [12] and is used in the present paper.

We refer to [9] and [13] for the theory of Schur functions (a brief account of Schur functions applied to Thom polynomials, can be found in [11, Sect. 3]). We shall use a notation for partitions indicating the number of times each integer occurs as a part. For example, we shall denote the partitions $(1,2,2,3)$, $(1,1,1,2,2)$, and $(1,1,1,1,1,2)$ respectively by 12^23, 1^32^2, and 1^52.

Using the theory of supersymmetric functions (cf., e.g., [13]), the Thom-Damon theorem can be rephrased by saying that there exist $\alpha_I \in \mathbf{Z}$ such that

$$\mathcal{T}^\Sigma = \sum_I \alpha_I S_I(R_m^* - R_n^*)\,, \tag{5}$$

the sum is over partitions I with $|I| = \operatorname{codim}(\Sigma)$. The expression in Eq. (5) is unique *(loc.cit.)*.

Example 1 *Let $S_I = S_I(R_m^* - R_n^*)$. We have the following three formulas valid for $m - n = 0, 1, 2$, respectively:*

$$\mathcal{T}^{A_4} = 24S_4 + 26S_{13} + 10S_{2^2} + 9S_{1^22} + S_{1^4}\,,$$
$$\mathcal{T}^{III_{2,3}} = 8S_{35} + 4S_{134} + 2S_{23^2}\,,$$
$$\mathcal{T}^{I_{2,2}} = S_{24^2} + 3S_{145} + 7S_{46} + 3S_{5^2}\,.$$

Note that all the coefficients in the formulas in Example 1 are nonnegative. For a more extensive list of examples, also for maps $\mathbf{C}^2 \to \mathbf{C}$, see the last section.

4 Cone classes for ample vector bundles

In the proof of our main result, we shall use the following two results of Fulton and Lazarsfeld from [5]. Recall first some classical definitions and facts from [6] (we shall also follow the notation from this book). Let E be a vector bundle of rank e on X. By a *cone* in E we mean a subvariety of E which is stable under the natural $\mathbb{G}_{\mathrm{m}}$-action on E. If $C \subset E$ is a cone of pure dimension d, then one may intersect its cycle $[C]$ with the zero-section of the vector bundle:

$$z(C,E) := s_E^*([C]) \in A_{d-e}(X)\,, \tag{6}$$

where $s_E^* : A_d(E) \to A_{d-e}(X)$ is the Gysin map determined by the zero section $X \to E$. For a projective variety X, there is well defined *degree* $\int_X : A_0(X) \to \mathbf{Z}$. We recall first the key technical result of [5].

Theorem 2 ([5, Theorem 2.1]) *Let E be an ample vector bundle of rank e on a projective variety X of dimension e, and let $C \subset E$ be a cone of pure dimension e. Then we have*

$$\int_X z(C,E) > 0.$$

Under the assumptions of the theorem, we also have in $H_0(X,\mathbf{Z})$ the homology analog of $z(C,E)$, denoted by the same symbol, and the homology degree map $\deg_X : H_0(X,\mathbf{Z}) \to \mathbf{Z}$. They are compatible with their Chow group counterparts via the cycle map: $A_0(X) \to H_0(X,\mathbf{Z})$ (cf. [6, Chap. 19]). Thus we have

$$\deg_X\big(z(C,E)\big) > 0\,. \tag{7}$$

Let P be a symmetric polynomial in e variables and of degree n. It has a unique presentation as a $\mathbf{Z}$-linear combination

$$\sum_I \beta_I \mathcal{S}_I\,,$$

where $|I| = n$, and $\mathcal{S}_I$ is the classical Schur function (cf. [9]). Recall that $\mathcal{S}_I$ is defined, e.g., by the $e \times e$ determinant (2) with a's replaced by the given variables, and all b's equal to zero. For any vector bundle E of rank e, we define $P(E)$ to be the image of P under the homorphism which sends the variables to the Chern roots of E. In other words,

$$P(E) = \sum_I \beta_I S_I(E)\,.$$

One says that P is *numerically positive for ample vector bundles* if for every projective variety X of dimension n, and every ample vector bundle E of rank e on X,

$$\int_X P(E)$$

is strictly positive. The second result characterizes polynomials numerically positive for ample vector bundles, with the help of Schur functions.

Theorem 3 ([5, Theorem I]) *A homogeneous polynomial*

$$\sum_I \beta_I \mathcal{S}_I ,$$

where $\beta_I \in \mathbf{Z}$, is numerically positive for ample vector bundles iff for any partition I we have $\beta_I \geq 0$, and additionally $\sum_I \beta_I > 0$.

For example, the nth Chern class $\mathcal{S}_{1^n}$ is numerically positive provided $n \leq e$, and in the surface case, i.e. for $n = 2$, any polynomial numerically positive for ample vector bundles is either a positive integer multiple of S_1, or a $\mathbf{Z}$-linear combination

$$aS_2 + bS_{1^2} ,$$

where $a, b \geq 0$ and $a + b > 0$ (these results are due respectively to Bloch-Gieseker and Kleiman, cf. the references in [5]).

5 Main result

We start with some preliminaries that are known in topology in a much more general framework. We use the notation from Section 2. We first pull back the bundle $\widetilde{\mathcal{J}}$ from BG to $BGL_m \times BGL_n$ via the map induced by the embedding

$$GL_m \times GL_n \hookrightarrow \mathrm{Aut}_m \times \mathrm{Aut}_n .$$

Since $GL_m \times GL_n$ acts linearly on $\mathcal{J}$, the obtained pullback bundle is now the vector bundle on $BGL_m \times BGL_n$ associated with the representation of $Gl_m \times GL_n$ on $\mathcal{J}$:

$$\mathcal{J}(R_m, R_n) := \left(\oplus_{i=1}^k \mathrm{Sym}^i(R_m^*)\right) \otimes R_n .$$

The bundle $\mathcal{J}(R_m, R_n)$ contains the preimage of $\widetilde{\Sigma} \subset \widetilde{\mathcal{J}}$, denoted by $\Sigma(R_m, R_n)$, whose dual class is given by the RHS of Eq. (5).

Consider, more generally, a pair of vector bundles E and F of ranks m and n on a variety X. We define the following vector bundle on X:

$$\mathcal{J}(E, F) := \left(\oplus_{i=1}^k \mathrm{Sym}^i(E^*)\right) \otimes F .$$

In fact, the pair of bundles (E, F) corresponds to a principal $GL_m \times GL_n$-bundle $B(E, F)$ and

$$\mathcal{J}(E, F) = B(E, F) \times_{GL_m \times GL_n} \mathcal{J}$$

is the bundle associated with the representation. Similarly, we define the singularity set

$$\Sigma(E,F) := B(E,F) \times_{GL_m \times GL_n} \Sigma \subset \mathcal{J}(E,F)\,.$$

The dual class[6] of $[\Sigma(E,F)]$ in

$$H^{2\,\mathrm{codim}(\Sigma)}(\mathcal{J}(E,F),\mathbf{Z}) \cong H^{2\,\mathrm{codim}(\Sigma)}(X,\mathbf{Z})$$

is equal to

$$\sum_I \alpha_I S_I(E^* - F^*)\,, \tag{8}$$

where the α_I's were defined in Eq. (5). The argument for that is fairly standard but one has to pass to topological homotopy theory, where each pair of bundles can be pulled back from the universal pair (R_m, R_n) of bundles on $BGL_m \times BGL_n$. It is possible to work entirely with the algebraic varieties. One can use the Totaro construction and representability for affine varieties ([16, proof of Theorem 1.3]).

The main result of the present paper, suggested/conjectured in [11], [12], and in [4] for Thom-Boardman singularities, is

Theorem 4 *Let Σ be a stable, nontrivial class of singularities. Then the Thom polynomial $\mathcal{T}^\Sigma$ is nonzero, and for any partition I the coefficient α_I in the Schur function expansion of the Thom polynomial $\mathcal{T}^\Sigma$ (cf. Eq. (5)) is nonnegative.*

Proof. We follow the notation from the first part of this section. Let $c = \mathrm{codim}(\Sigma)$ (so that for any partition I with $\alpha_I \neq 0$, we have $|I| = c$). Suppose that X is a projective variety of dimension c, F is an ample vector on X with $\mathrm{rank}(F) = n' = n + r \geq c$, and $E = \mathbf{1}_X^{m'}$ is a trivial bundle of rank $m' = m + r$ (for some $r \geq 0$). The variety $\Sigma(\mathbf{1}^{m'}, F)$ is a cone in $\mathcal{J}(\mathbf{1}^{m'}, F)$ because $\mathbb{G}_\mathrm{m} \subset \mathrm{Aut}_{n'}$. From the first part of this section, we know that the cone class

$$z\big(\Sigma(E,F), \mathcal{J}(E,F)\big) \in H_0(X,\mathbf{Z})$$

is dual to the universal expression (8):

$$\sum_I \alpha_I S_I(E^* - F^*) = \sum_I \alpha_I S_I(-F^*) = \sum_I \alpha_I S_{I^\sim}(F)$$

(we use here Eqs. (2) and (4)). Using the notation of Theorem 3, we set

$$P = \sum_I \alpha_I \mathcal{S}_{I^\sim}\,.$$

[6]A precise definition of the dual class of $[\Sigma(E,F)]$, in the case when X is singular, is given in Note 6.

Since a direct sum of ample vector bundles is ample [7, Proposition 2.2], the vector bundle

$$\mathcal{J}(\mathbf{1}^{m'},F)=F^{\oplus N}$$

(for some integer N) is ample. Hence we have we have by the inequality (7)

$$\int_X P(F)=\deg_X\bigl(z(\Sigma(\mathbf{1}^{m'},F),F^{\oplus N})\bigr)>0\,,$$

that is, the polynomial P is numerically positive for ample vector bundles. We conclude, by Theorem 3, that the coefficients α_I in Eq. (5) are nonnegative. Moreover, we have $\sum_I \alpha_I>0$, which implies that $\mathcal{T}^\Sigma\neq 0$.

The theorem has been proved. □

Theorems 2 and Theorem 3 are valid over any algebraically closed field of arbitrary characteristic (cf. [5]). Therefore our main result would hold in that generality provided that one develops a suitable theory of Thom polynomials. Of course, usual cohomology should be replaced by Chow rings, cf. [16]. By Theorem 1.3 *(loc.cit.)*, the theory of characteristic classes is reduced to the calculus in the Chow ring of a classifying space.

Remark 5 The functor of k-jets

$$E,F\mapsto\mathcal{J}(E,F)=\Bigl(\oplus_{i=1}^k \mathrm{Sym}^i E^*\Bigr)\otimes F$$

and the cone $\Sigma(E,F)$ can be replaced by more general construction: an arbitrary functor $\phi(E,F)$ and a cone bundle $\Sigma(E,F)\subset\phi(E,F)$. We assume the following two properties:

1. the class $z(\Sigma(E,F),\phi(E,F))$ is stable under simultaneous addition of the same bundle;

2. the assignment $F\mapsto\phi(\mathbf{1}^m,F)$ (or $E\mapsto\phi(E^*,\mathbf{1}^n)$) preserves ampleness.

Examples of functors preserving ampleness for fields of characteristic zero are *polynomial functors*. They are, at the same time, quotient functors and subfunctors of the tensor power functors (cf. [7]). Theorem 4 remains then valid.

Note 6 We give here a precise definition of the dual class of $[\Sigma(E,F)]$ for possibly singular X. This class will lie in $H^{2c}(X,\mathbf{Z})$ (recall that $c=\mathrm{codim}(\Sigma)$). Let $Y\subset\Sigma$ be the set of singular points of Σ, i.e.

$$\Sigma\setminus Y\subset\mathcal{J}\setminus Y$$

is a submanifold. The set Y is $GL_m \times GL_n$-invariant. For a pair of bundles we construct the fibering

$$Y(E,F) = B(E,F) \times_{GL_m \times GL_n} Y \subset \Sigma(E,F) \subset \mathcal{J}(E,F)$$

with fiber Y. We have $\operatorname{codim}_{\mathbf{C}} Y \geq c+1$ and therefore

$$H^i(\mathcal{J} \setminus Y, \mathbf{Z}) \cong H^i(\mathcal{J}, \mathbf{Z})$$

for $i \leq 2c$. Hence

$$H^i(\mathcal{J}(E,F) \setminus Y(E,F), \mathbf{Z}) \cong H^i(\mathcal{J}(E,F), \mathbf{Z})$$

in the same range of degrees, and it is enough to define the desired class in $H^{2c}(\mathcal{J}(E,F) \setminus Y(E,F), \mathbf{Z})$. Note that $\Sigma(E,F) \setminus Y(E,F)$ has a normal bundle. Its Thom class defines an element in

$$H^{2c}(\mathcal{J}(E,F) \setminus Y(E,F), \mathcal{J}(E,F) \setminus \Sigma(E,F); \mathbf{Z})\,.$$

The image of this element in

$$H^{2c}(\mathcal{J}(E,F) \setminus Y(E,F), \mathbf{Z}) \cong H^{2c}(\mathcal{J}(E,F), \mathbf{Z}) \cong H^{2c}(X, \mathbf{Z})$$

is the desired class. Verification that this class is natural with respect to pull back of vector bundles is left to the reader.

6 Examples

Let $S_I = S_I(R_m^* - R_n^*)$ in the notation of Section 2. We list several examples of singularities $\Sigma[n-m] : M \to N$, where $m = \dim(M) \leq n = \dim(N)$. All of them were computed by the "method of restriction equations" of [14]. In the first paragraph we give the Schur function expansions of Thom polynomials for singularities $\Sigma[0]$ with codimension ≤ 6 from [14], p. 508. In the second paragraph, we give the Schur function expansion of the Thom polynomials for the singularities $\Sigma[1]$) from [14], p. 512. In the third paragraph, we list some examples from [11], [12]. (In [11], [12] we used a "shifted" notation: $\Sigma(n-m+1)$.)

$A_1[0]$: S_1
$A_2[0]$: $2S_2 + S_{1^2}$
$A_3[0]$: $6S_3 + 5S_{12} + S_{1^3}$
$A_4[0]$: $24S_4 + 26S_{13} + 10S_{2^2} + 9S_{1^22} + S_{1^4}$
$I_{2,2}[0]$: S_{2^2}
$A_5[0]$: $120S_5 + 154S_{14} + 92S_{23} + 71S_{1^23} + 14S_{1^32} + 35S_{12^2} + S_{1^5}$
$I_{2,3}[0]$: $4S_{23} + 2S_{12^2}$
$A_6[0]$: $720S_6 + 1044S_{15} + 770S_{24} + 266S_{3^2} + 580S_{1^24} + 455S_{123} + 70S_{2^3} + 155S_{1^33} + 84S_{1^22^2} + 20S_{1^42} + S_{1^6}$

$I_{2,4}[0]$: $16S_{24} + 4S_{3^2} + 12S_{123} + 5S_{2^3} + 2S_{1^22^2}$
$I_{3,3}[0]$: $2S_{24} + 6S_{3^2} + 3S_{123} + S_{1^22^2}$

$A_1[1]$: S_2
$A_2[1]$: $4S_4 + 2S_{13} + S_{2^2}$
$A_3[1]$: $36S_6 + 30S_{15} + 19S_{24} + 5S_{3^2} + 6S_{1^24} + 5S_{123} + S_{2^3}$
$A_4[1]$: $507S_8 + 555S_{17} + 391S_{26} + 240S_{35} + 76S_{4^2} + 216S_{1^26} + 210S_{125} + 104S_{134} + 55S_{2^24} + 21S_{23^2} + 24S_{1^35} + 26S_{1^224} + 10S_{1^23^2} + 9S_{12^23} + S_{2^4}$
$III_{2,2}[1]$: S_{3^2}
$I_{2,2}[1]$: $3S_{34} + S_{13^2}$
$III_{2,3}[1]$: $8S_{35} + 4S_{134} + 2S_{23^2}$

$I_{2,2}[1]$: $S_{13^2} + 3S_{34}$
$I_{2,2}[2]$: $S_{24^2}+3S_{145}+7S_{46}+3S_{5^2}$
$I_{2,2}[3]$: $S_{35^2}+3S_{256}+7S_{157}+3S_{16^2}+15S_{58}+10S_{67}$

In this paragraph, we list Thom polynomials for the functions $\mathbf{C}^2 \to \mathbf{C}$ of the singularities A, D, E with codimension at most 6, computed by the "method of restriction equations" of [14]. The results agree with those in [8] given in another basis.

A_1: S_{1^2}
A_2: $2S_{1^3} + 2S_{12}$
A_3: $5S_{1^4} + 11S_{1^22} + 6S_{2^2} + 6S_{13}$
A_4: $12S_{1^5} + 44S_{1^32} + 44S_{12^2} + 56S_{1^23} + 36S_{23} + 24S_{14}$
D_4: $S_{1^5} + 3S_{1^32} + 6S_{12^2} + 2S_{1^23} + 4S_{23}$
A_5: $30S_{1^6} + 160S_{1^42} + 248S_{1^22^2} + 338S_{1^33} + 434S_{123} + 328S_{1^24} + 108S_{3^2} + 228S_{24} + 120S_{15}$
D_5: $4S_{1^6} + 18S_{1^42} + 42S_{1^22^2} + 26S_{1^33} + 64S_{123} + 12S_{1^24} + 24S_{3^2} + 24S_{24}$
A_6: $79S_{1^7}+566S_{1^52}+1238S_{1^32^2}+1723S_{1^43}+3473S_{1^223}+2736S_{1^34}+1834S_{13^2}+3898S_{124} + 2220S_{1^25} + 1260S_{34} + 1632S_{25} + 720S_{16}$
D_6: $8S_{1^7} + 50S_{1^52} + 138S_{1^32^2} + 118S_{1^43} + 348S_{1^223} + 124S_{1^34} + 224S_{13^2} + 320S_{124} + 48S_{1^25} + 144S_{34} + 96S_{25}$
E_6: $3S_{1^7}+18S_{1^52}+54S_{1^32^2}+39S_{1^43}+129S_{1^223}+36S_{1^34}+102S_{13^2}+102S_{124}+12S_{1^25} + 60S_{34} + 24S_{25}$

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